ON TRANSVERSELY FLAT CONFORMAL FOLIATIONS WITH GOOD MEASURES

TARO ASUKE

ABSTRACT. Transversely flat conformal foliations with good transverse invariant measures are Riemannian in the $C^{1+\mathrm{Lip}}$ sense. In particular, transversely similar foliations with good measures are transversely Riemannian as transversely C^{ω}-foliations.

INTRODUCTION

The study of foliations with some transverse structure has attracted many people, and there are a lot of results for certain kinds of these foliations, such as Riemannian foliations, transversely affine foliations, and so on. In particular, transversely Riemannian foliations have been deeply studied, and we have the structure theorem (Molino [10]).

As a natural generalization, we can consider the transversely conformal foliations, which has been studied by Blumenthal [3], Vaisman [13], and others. This class is known to be quite large, so that all codimension one or transversely holomorphic foliations are transversely conformal.

Now we restrict our attention to foliations with transverse flat conformal structures. As is well-known, the flat conformal geometry appears as the geometry at infinity of the hyperbolic geometry.

It is well-known that certain kinds of flat conformal manifolds admit a Riemannian metric compatible with its flat conformal structure. This suggests that we may expect that, under some conditions, there exists a transverse invariant Riemannian metric for transversely flat conformal foliations, that is, the foliation is Riemannian.

In this paper we study transversely flat conformal foliations admitting non-atomic, full-support transverse invariant measures, which we call good measures, and obtain the following results.

Theorem A. *Let $(M, \mathcal{F})$ be a transversely flat conformal foliation of a closed manifold M. Assume that there is a good measure μ on M. Then there is a transverse invariant Riemannian metric g of $(M, \mathcal{F})$ which is of class $C^{1+\mathrm{Lip}}$, namely, of class C^1 with the Lipschitz derivative.*

The proof of Theorem A proceeds as follows. First, in section 3, we use slightly generalized versions of Ghys' lemmas [6] to show that the leaf space $\widehat{M}$ of the

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universal covering of M equipped with the lifted foliation of $\mathcal{F}$ is a Hausdorff manifold. Then in section 4 we apply the theorem of Kulkarni-Pinkall [8] which ensures the existence of a Riemannian metric of class $C^{1+\mathrm{Lip}}$ under the action of the full group $\mathrm{Conf}(\widehat{M})$ of conformal automorphisms as a Möbius manifold.

Remark. We do *not* know any example which satisfies all the assumptions of theorem A but is not a Riemannian foliation as a transversely C^ω-foliation.

Now we assume that the foliation is of dimension one and all orbits are dense. For any flows on a closed manifold, there is a non-trivial transverse invariant ergodic measure μ. The support of μ must be the whole manifold, and μ is of course non-atomic. Thus μ is good. So the theorem applies, and we obtain the following.

Corollary B. *If $(M, \mathcal{F})$ is a transversely flat conformal flow of a closed manifold with dense orbits, then $(M, \mathcal{F})$ is transversely Riemannian in the $C^{1+\mathrm{Lip}}$ sense.*

In section 3, we define the holonomy group of $(M, \mathcal{F})$. If the holonomy group is contained in $\mathrm{Sim}(\mathbf{R}^q)$, which is the case if the foliation is transversely similar, we have stronger results.

Theorem C. *Let $(M, \mathcal{F})$ be a transversely flat conformal foliation of a closed manifold M with a good measure. Assume that the holonomy group is contained in $\mathrm{Sim}(\mathbf{R}^q)$. Then $(M, \mathcal{F})$ is a Riemannian foliation as a transversely C^ω foliation.*

In the paper [9] Matsumoto showed a similar result for flat conformal manifolds. To show Theorem C, we will generalize the method of Fried [5] and Matsumoto [9] to foliations under the existence of good measures. This step is done in section 5.

In Carrière [4] he showed the following theorem.

Theorem. *Let $(M, \mathcal{F})$ be a transversely Riemannian flow of a closed manifold with dense orbits. We assume that $M, \mathcal{F}$ are orientable and that $M, \mathcal{F}$, and the bundle-like metric on M are of class C^r, $r \geq 2$. Then $(M, \mathcal{F})$ is differentiably conjugate to an irrational linear flow on the n-dimensional torus T^n. That is, there is a diffeomorphism from M to T^n which maps orbits to orbits.*

Now let $(M, \mathcal{F})$ be a transversely flat conformal flow of a closed manifold with dense orbits. We assume that the holonomy group is contained in $\mathrm{Sim}(\mathbf{R}^q)$. Then by the same argument as in the proof of Corollary B, Theorem C applies in this case. Combining with the above theorem, we have the following.

Theorem D. *Let $(M, \mathcal{F})$ be a transversely flat conformal flow of a closed manifold M with all orbits dense. Suppose that $M, \mathcal{F}$ are of class C^r, $r \geq 2$, and orientable, and that the holonomy group is contained in $\mathrm{Sim}(\mathbf{R}^q)$. Then $(M, \mathcal{F})$ is C^r-conjugate to an irrational linear flow on the n-dimensional torus T^n in the above sense.*

In the paper [2], the author will classify the transversely flat conformal flows with good measures, which will show that under the hypothesis of Theorem D, the flow is in fact an $(\mathrm{Isom}(\mathbf{R}^q), \mathbf{R}^q)$ flow. Notice that Theorem D is a complete analogue to the theorem of Carrière and that Theorems C and D are partial extention of the results by Nishimori [11] and Ghys [6].

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1. Definitions

Throughout this paper, we work in the C^r-category, $r \geq 1$, unless otherwise stated. In the following, $\widehat{\mathbf{R}^n}$ $(= \mathbf{R}^n \cup \{\infty\} \cong S^n)$ denotes the one point compactification of $\mathbf{R}^n$, and $\mathbf{R}^+$ the positive real numbers.

Definition 1.1. We denote by $\mathrm{Sim}(\mathbf{R}^q)$ the group of similarity transformations of $\mathbf{R}^q$, namely, the group of transformations of $\mathbf{R}^q$ of the form $x \mapsto r \cdot Ax + v$, where $r \in \mathbf{R}^+$, $A \in \mathrm{O}(q)$, and v is a vector in $\mathbf{R}^q$, and we denote by $\mathrm{Conf}(\widehat{\mathbf{R}^q})$ the group of Möbius transformations of $\widehat{\mathbf{R}^q}$, which is by definition the group generated by $\mathrm{Sim}(\mathbf{R}^q)$ and the inversion $J(x) = x/\|x\|^2$ with respect to the unit sphere. Notice that we can naturally consider $\mathrm{Sim}(\mathbf{R}^q) \subset \mathrm{Conf}(\widehat{\mathbf{R}^q})$.

Notice that any conformal automorphism of S^q is a Möbius transformation.

Let M be an n-dimensional closed manifold and $\mathcal{F}$ a q-codimensional foliation of M. We say $(M, \mathcal{F})$ is transversely flat conformal if $(M, \mathcal{F})$ is a $(\mathrm{Conf}(S^q), S^q)$ foliation, namely, if there are an open covering $\{U_i\}_{i \in I}$ of M and a family $\{f_i : U_i \to S^q\}$ of submersions such that
   1) $\{(U_i, f_i)\}$ defines $\mathcal{F}$, i.e., $U_i \cong O_i \times T_i$ and $\mathcal{F}|_{U_i} = \{O_i \times \{p\}; p \in T_i\}$, where O_i is an open set in $\mathbf{R}^{n-q}$, $T_i = f_i(U_i)$, and
   2) the transition functions $\gamma_{ji} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are elements of $\mathrm{Conf}(S^q)$.

Here we may assume by decomposing the open set U_i that $\overline{T_i} \neq S^q$, and that the index set I is a finite set because M is compact. Moreover we may assume that
   3) Each T_i is an open ball in $\mathbf{R}^q$.

So by changing f_i, we may assume
   4) The closures $\overline{T_i}$ of T_i are disjoint.

We say $(M, \mathcal{F})$ is transversely similar if $(M, \mathcal{F})$ is a $(\mathrm{Sim}(\mathbf{R}^q), \mathbf{R}^q)$ foliation.

Remark. Any transversely flat conformal foliation is transversely of class C^ω by definition.

We now put $T = \amalg\, T_i$, and Γ denotes the pseudogroup associated with T. Since M is compact, we can take concentric open balls T_i' in T_i such that $\overline{T_i'}$ is contained in T_i and the pseudogroup Γ' associated with $T = \amalg\, T_i'$ is equivalent to Γ.

By definition, T and T' are contained in $\mathbf{R}^q$, and at the same time, we may regard them as submanifolds of M in the natural way.

Definition 1.2. A transverse invariant measure μ of $(M, \mathcal{F})$ is said to be good if μ has the following properties.
   a) $\mathrm{supp}\,\mu = M$, and
   b) μ is non-atomic.
We also assume that μ is a Borel measure, and that μ is locally finite.

Notice that the required properties for measures are somehow natural, because every Riemannian foliation has such a measure even if we do not assume the differentiability.

Hereafter we always assume that $(M, \mathcal{F})$ has a good measure, and that $(M, \mathcal{F})$ is oriented and transversely oriented for simplicity. Thus when we consider transformation groups, we always assume that they consist of orientation preserving elements.

2. Examples

Example 2.1. Let M be a closed manifold of constant negative curvature and $\mathcal{F}$ the geodesic flow on the unit tangent bundle UM of M. Then $(UM, \mathcal{F})$ has a dense orbit and a good measure but does not admit any transverse flat conformal structure. It is never Riemannian.

Example 2.2. Consider S^2, the unit sphere in $\mathbf{R}^3$, and define γ_1 and γ_2 as follows.

First, let B be the unit ball in $\mathbf{R}^3$ and put $C_1 = B \cap \{y = z = 0\}$ and $C_2 = B \cap \{x = 0 \text{ and } (y - \sqrt{2})^2 + z^2 = 1\}$. Then we consider the Poincaré model on B and define γ_1 and γ_2 to be the irrational rotations whose axes are C_1 and C_2, respectively.

Now let Σ be a closed surface of genus two, Γ the group generated by γ_1 and γ_2, and $(M, \mathcal{F})$ the suspension of (S^2, Γ) over Σ. Then all leaves of $\mathcal{F}$ are dense and $(M, \mathcal{F})$ is transversely flat conformal but does not admit any good measure. It is easy to see that $(M, \mathcal{F})$ cannot be Riemannian.

This example shows that Corollary B cannot be generalized to higher dimensional foliations.

Example 2.3. Consider $\mathbf{R}^2 = \mathbf{C}$ and the standard lattice $\mathbf{Z}^2$ in $\mathbf{C}$. We define $\gamma : \mathbf{C} \to \mathbf{C}$ by $\gamma(x, y) = (x + \theta, y + \varphi)$ and let Γ be the group generated by γ. Here we consider $\mathbf{C}/\mathbf{Z}^2 = T^2$, the 2-dimensional torus.

Now we transport everything by the exponential map to $\mathbf{C}\backslash\{0\}$, and consider the suspension $(M, \mathcal{F})$ of (T^2, Γ) over S^1. Then $(M, \mathcal{F})$ satisfies all the assumptions of Theorem C, and is transversely Riemannian; in fact, $(M, \mathcal{F})$ is differentiably conjugate to a linear flow on the flat torus.

In particular, for suitable choices of θ and φ, all orbits of γ are dense in T^2. Then Theorem D applies, and again we can see the foliation is transversely Riemannian as a transversely C^ω foliation.

Example 2.4. Consider the Hopf fibration $S^3 \to S^2$ and let X be a vector field tangent to the fibres. We choose X appropriately so that if we consider its 1-parameter group φ_t, then $\{\varphi_n(x)\}_{n \in \mathbf{Z}}$ is dense in the fibre passing through x.

Now we put $X = S^3 \times (0, \infty)$, and define φ, ψ as follows,

$$\varphi(x, t) = (\varphi_1(x), rt) \quad \text{and} \quad \psi(x, t) = (x, 2t).$$

Notice that if we consider S^3 as the unit sphere in $\mathbf{C}^2$, then we have

$$\varphi_1(z_1, z_2) = (e^{2\pi\sqrt{-1}\theta} z_1, e^{2\pi\sqrt{-1}\theta} z_2),$$

where θ is an irrational number. Therefore φ is an orthogonal transformation.

Then we put $N = X/\langle \psi \rangle$ and consider the suspension $(M, \mathcal{F})$ of (N, f), where f is the automorphism induced by φ.

It is clear that $(M, \mathcal{F})$ is a $(\mathrm{CO}^+(4), \mathbf{R}^4 \setminus \{0\})$ flow, where $\mathrm{CO}^+(4) = \{h \in \mathrm{Sim}(\mathbf{R}^4); h(0) = 0\}$. We equip $\mathbf{R}^4 \setminus \{0\}$ with the metric defined by the formula

$$g_x = \frac{1}{\|x\|^2} g_E,$$

where g_E denotes the Euclidean metric and $\|\cdot\|$ denotes the Euclidean norm.

It is easy to see the action of $\mathrm{CO}^+(4)$ preserves this metric, thus $(M, \mathcal{F})$ is a Riemannian flow.

3. Preliminaries

First of all, notice that since T and T' are contained in $\mathbf{R}^q$, we have a natural Riemannian metric on T and also on T' induced by the Euclidean metric on $\mathbf{R}^q$. We denote by $D_\eta(x)$ the open ball of radius η and centered at x, and by $\operatorname{rad} D$ the radius of a ball D.

The following two lemmas appeared in Ghys [6].

Lemma 3.1. *For any positive real number ϵ, there is a positive real number η with the following property:*

If an element $\gamma \in \Gamma$ is defined on $D_\eta(x)$ and satisfies $\gamma x \in \overline{T'}$, where $x \in T'$, then $\operatorname{rad} \gamma D_\eta(x)$ is less than ϵ.

Proof. First we claim that for any positive real number δ, there is a positive real number η such that $\mu(D_\eta) < \delta$ holds for any $D_\eta \subset T$ satisfying $D_\eta \cap \overline{T'} \neq \phi$. For otherwise, we can find a positive real number δ and a sequence $\{B_n\}$ of open balls such that $\operatorname{rad} B_n = 1/n$, $B_n \cap \overline{T'} \neq \phi$, and $\mu(B_n) \geq \delta$. Then we may assume B_n converges to some point y in $\overline{T'}$, and y is an atom of μ. This contradicts the assumption that μ is non-atomic.

Secondly we show that for any positive real number ϵ, there is a positive real number δ such that $\operatorname{rad} D < \epsilon$ holds for any open ball D in T satisfying $\mu(D) < \delta$ and $D \cap \overline{T'} \neq \phi$. For otherwise, we can find a positive real number ϵ and a sequence $\{B_n\}$ of open balls satisfying $\mu(B_n) < 1/n$, $B_n \cap \overline{T'} \neq \phi$, and $\operatorname{rad} B_n = \epsilon$. Now we may assume ϵ is less than the half of the distance between ∂T and $\partial T'$ in $\mathbf{R}^q$; then we may also assume that B_n converges to some open ball B in T. If we take an open ball B' such that $\overline{B'} \subset B$, then we have $B' \subset B_n$ for sufficiently large n. So we have $\mu(B') \leq \mu(B_n) < 1/n$, and hence $\mu(B') = 0$ holds. This contradicts the assumption that μ is full-support.

Finally we show that for any positive real number ϵ, there is a positive real number η such that $\operatorname{rad} B_1 < \eta$ implies $\operatorname{rad} B_2 < \epsilon$ for any pair of open balls B_1 and B_2 in T satisfying $B_i \cap \overline{T'} \neq \phi$, $i = 1, 2$, and $\mu(B_1) = \mu(B_2)$. For the given positive real number ϵ, choose δ as in the second claim and choose η as in the first claim for this δ. Then if $\operatorname{rad} B_1 < \eta$, we have $\mu(B_2) = \mu(B_1) < \delta$ and consequently $\operatorname{rad} B_2 < \epsilon$.

Now we prove Lemma 3.1. Since γ maps balls to balls, preserves μ, and $\gamma D_\eta(x) \cap \overline{T'} \ni \gamma x$, by putting $B_1 = D_\eta(x)$ and $B_2 = \gamma D_\eta(x)$, the above assertion applies and we complete the proof. □

Lemma 3.2. *There is a positive real number δ_0 such that the germ γ'_x of an element of Γ' at any point x in T' extends to an element of Γ defined on $D_{\delta_0}(x)$.*

Proof. Let Γ'_1 denote the finite set of the natural generators of Γ', and Γ'_k denote the composition of k elements of Γ'_1. Then there is a positive real number δ such that for any x in T' and the germ γ'_x of an element of Γ'_1 at x, γ'_x extends to $\gamma \in \Gamma$ defined on $D_\delta(x)$. By Lemma 3.1, there is a positive real number $\delta_0 \leq \delta$ such that $\operatorname{rad} \gamma D_{\delta_0}(x) < \frac{\delta}{2}$ holds for all $\gamma \in \Gamma$ defined on $D_{\delta_0}(x)$ with $\gamma x \in T'$.

Now by the choice of δ_0, Lemma 3.2 is true for Γ'_1, so we assume that Lemma 3.2 is true for Γ'_k and prove it for Γ'_{k+1}.

Let $\gamma \in \Gamma'_{k+1}$ and write $\gamma = \gamma_1 \gamma_2$, where $\gamma_1 \in \Gamma'_1$ and $\gamma_2 \in \Gamma'_k$. By the induction hypothesis, γ_2 is defined on $D_{\delta_0}(x)$ and $\gamma_2 D_{\delta_0}(x) \subset D_\delta(\gamma_2 x)$ holds. Since γ_1 is defined on $D_\delta(\gamma_2 x)$, $\gamma = \gamma_1 \gamma_2$ is also defined on $D_{\delta_0}(x)$. □

Now we consider the universal covering $\widetilde{M}$ of M and let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{M}$. Then we have the holonomy homomorphism $\phi : \pi_1(M) \to \mathrm{Conf}(S^q)$ and the developing map $D : \widetilde{M} \to S^q$. We denote by $H = \phi(\pi_1(M))$ the holonomy group, and let $\widehat{M} = \widetilde{M}/\widetilde{\mathcal{F}}$. We denote by p the natural projection from $\widetilde{M}$ to $\widehat{M}$.

Lemma 3.3. *The leaf space $\widehat{M}$ is a Hausdorff manifold of dimension q.*

Proof. It is well-known that $\widehat{M}$ is a possibly non Hausdorff manifold of dimension q. So it suffices to show that $\widehat{M}$ is Hausdorff. Suppose the contrary; then there are two points x and y of $\widetilde{M}$ belonging to distinct leaves of $\widetilde{\mathcal{F}}$, and two sequences $\{x_n\}$ and $\{y_n\}$ of $\widetilde{M}$ such that x_n and y_n belong to the same leaf for each n and x_n converges to x and y_n converges to y. We choose lifts $\widetilde{T}_i$ of T_i and $\widetilde{T}'_i$ of T'_i embedded in $\widetilde{M}$, respectively, and we may assume that x, x_n are in $\widetilde{T}'_i$ and y, y_n are in $\widetilde{T}'_j$ for some integers i and j.

If we induce a metric to $\widetilde{T}'_i$ from T'_i for each i, then clearly the conclusions of Lemmas 3.1 and 3.2 are true for the pseudogroups $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ of $\widetilde{\mathcal{F}}$ associated with $\widetilde{T}$ or $\widetilde{T}'$.

Now we may assume $d(x_n, x) < \delta_0$ for all n; then any element $\widetilde{\gamma}$ of $\widetilde{\Gamma}'$ which maps x_1 to y_1 is defined on $D_{\delta_0}(x_1)$, which contains x and x_n for any n. Noticing that the developing map D is one to one on each $\widetilde{T}_i$ by definition, we see that y_n is the only point which lies on the same leaf as x_n, or $\widetilde{\gamma}(x_n)$. Consequently we have $y_n = \widetilde{\gamma}(x_n)$ for each n, and hence $y = \widetilde{\gamma}(x)$. This is a contradiction, because we assumed that x and y are not in the same leaf. $\qquad\square$

4. Proof of Theorem A

We use the same notation as in the previous sections.

We have the action of $\pi_1(M)$ on $\widetilde{M}$ which preserves the lifted foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M}$. So $\pi_1(M)$ naturally acts on $\widehat{M}$. On the other hand, the developing map D obviously projects down to a mapping $\Delta : \widehat{M} \to S^q$, which is a local homeomorphism.

Now we have the action of $\pi_1(M)$ on S^q via ϕ. By definition of Δ, it is clear that Δ is equivariant with respect to the action of $\pi_1(M)$, i.e., $\Delta(\gamma(x)) = \phi(\gamma)(\Delta(x))$ holds for any $x \in \widehat{M}$ and $\gamma \in \pi_1(M)$.

If we find a Riemannian metric on $\widehat{M}$ which is invariant under the action of $\pi_1(M)$, then we have a transverse invariant metric on $\widetilde{M}$ which is invariant under the action of $\pi_1(M)$, and by projecting it down to M we obtain a transverse invariant Riemannian metric on M. So our goal is to find an invariant metric on $\widehat{M}$.

We divide the study into the following three cases.

1) Δ is a homeomorphism onto S^q.
2) Δ is a homeomorphism onto $\mathbf{R}^q$.
3) Otherwise.

We will show in the rest of this section that $(M, \mathcal{F})$ is transversely Riemannian as a transversely C^ω foliation in cases 1) and 2), $C^{1+\mathrm{Lip}}$ in case 3).

Remark. We will show in the next section that $(M, \mathcal{F})$ is a transversely C^ω Riemannian foliation if H is contained in $\mathrm{Sim}(\mathbf{R}^q)$.

Notice that $\widehat{M}$ inherits a locally finite measure $\widehat{\mu}$ which is non-atomic and satisfies $\mathrm{supp}\, \widehat{\mu} = \widehat{M}$, and is invariant under the action of $\pi_1(M)$.

First we treat case 1). We recall the notion of the limit sets of subgroups of $\mathrm{Conf}(S^q)$ and related facts. We consider $S^q = \widehat{\mathbf{R}^q}$ as the unit sphere in $\mathbf{R}^{q+1}$; then $\mathrm{Conf}(S^q)$ acts on the closed unit ball D^{q+1} in $\mathbf{R}^{q+1}$ in the standard way.

Definition 4.1. We define the limit set $\Lambda(H)$ of H in the same way as for the discrete groups, namely,

$$\Lambda(H) = \{x \in S^q; \, x = \lim_{n \to \infty} \gamma_n(p) \text{ for some sequence } \{\gamma_n\} \text{ of } H\},$$

where p is a point in D^{q+1}. It is easy to see that this definition is independent of the choice of the point p.

We say H is elementary if Λ consists of at most two points, non-elementary otherwise.

We use the following lemmas.

Lemma 4.2 ([9, Proposition 5.12]). *For a subgroup H of $\mathrm{Conf}(\widehat{\mathbf{R}^q})$, the following conditions are equivalent.*
  1) *H is precompact.*
  2) *$\Lambda(H)$ is empty.* □

Lemma 4.3 ([9, Proposition 5.13]). *If a subgroup H of $\mathrm{Conf}(S^q)$ is precompact, then H is conjugate to a subgroup of $\mathrm{SO}(q+1) \subset \mathrm{Conf}(S^q)$ by an element of $\mathrm{Conf}(S^q)$.* □

We denote by $\mathcal{L}$ the closure of the set of fixed points of loxodromic elements of H. Recall that we say an element h of $\mathrm{Conf}(S^q)$ is loxodromic if and only if h has two fixed points in S^q.

The following proposition is well-known.

Proposition 4.4. *If $\Lambda(H)$ consists of at least two points, then we have $\mathcal{L} = \Lambda(H)$.*

Proof. See Theorem 5.15 and Lemma 5.6 of [9]. □

Now we have the following lemma.

Lemma 4.5. *In case 1), H is conjugate to a subgroup of $\mathrm{SO}(q+1)$.*

Proof. In this case, we can define a finite measure $\overline{\mu}$ on S^q by projecting down $\widehat{\mu}$. Then $\overline{\mu}$ is invariant under the action of H, non-atomic, and $\mathrm{supp}\,\mu = S^q$. Consequently, any element of H is not loxodromic.

Now the above proposition shows that $\Lambda(H)$ consists of at most one point. So we assume that $\Lambda(H)$ consists of a single point. Then we may assume $\Lambda(H) = \{\infty\}$, and then H is contained in $\mathrm{Sim}(R^q)$. Since H preserves the good measure $\overline{\mu}$, any element of H belongs to $\mathrm{Isom}(\mathbf{R}^q)$. Thus we can write any element h of H as

$$h(x) = A_h x + v_h \text{ , where } A_h \in \mathrm{SO}(q) \text{ , and } v_h \in \mathbf{R}^q.$$

Now we consider the upper half space model. So we regard H as acting on $\overline{\mathbf{H}}^{q+1} = \{x = (x_1, \ldots, x_{q+1}) \in \mathbf{R}^{q+1}; \, x_q \geq 0\}$. We write an element h of H as $\widetilde{h}$ when we consider the action of H on $\overline{\mathbf{H}}^{q+1}$. Then for $x = (x_1, \ldots, x_{q+1}) \in \overline{\mathbf{H}}^{q+1}$, we have

$$\widetilde{h}(x_1, \ldots, x_{q+1}) = (A_h x' + v_h, x_{q+1}) \text{ , where } x' = (x_1, \ldots, x_q).$$

Since $\Lambda(H) = \{\infty\}$, we can find a sequence $\{h_i\}$ of elements of H such that

$$\lim_{i \to \infty} \widetilde{h}_i(0, \ldots, 0, 1) = \infty$$

holds. Then we have

$$\lim_{i \to \infty} h_i(B_1) = \{\infty\},$$

where B_1 denotes the open unit ball in $\mathbf{R}^q$. This implies ∞ is an atom of $\overline{\mu}$.

This is a contradiction, and we see that $\Lambda(H)$ is empty. Then Lemmas 4.2 and 4.3 prove the lemma. □

Lemma 4.6. *In case* 2)*, H is contained in* $\mathrm{Isom}(\mathbf{R}^q)$.

Proof. As in the first case, we can project down $\widehat{\mu}$ to $\mathbf{R}^q$ and obtain a locally finite measure $\overline{\mu}$ on $\mathbf{R}^q$ which is invariant under the action of H.

Since every element of H fixes the point ∞, we see that H is a subgroup of $\mathrm{Sim}(\mathbf{R}^q)$. Now $\overline{\mu}$ is non-atomic, so H is a subgroup of $\mathrm{Isom}(\mathbf{R}^q)$. □

Lemmas 4.5 and 4.6 show that the foliation is transversely Riemannian as a transversely C^ω foliation in cases 1) and 2).

Now we assume case 3) holds and show that there is an invariant Riemannian metric on $\widehat{M}$ of class $C^{1+\mathrm{Lip}}$. The only assumption we need is that $\widehat{M}$ is a Hausdorff manifold and Δ is neither a homeomorphism onto S^q nor $\mathbf{R}^q$. In fact, we prove the following theorem.

Theorem 4.7. *Let $(M, \mathcal{F})$ be a transversely flat conformal foliation of a closed manifold M. Assume that the leaf space of the lifted foliation $(\widetilde{M}, \widetilde{\mathcal{F}})$, where $\widetilde{M}$ denotes the universal covering, is a Hausdorff manifold. Assume also that the mapping Δ from $\widetilde{M}$ to S^q induced by the developing map D is a homeomorphism onto neither S^q nor $\mathbf{R}^q$. Then there is a transverse invariant Riemannian metric g of $(M, \mathcal{F})$ which is of class $C^{1+\mathrm{Lip}}$. In particular the metric g is twice differentiable almost everywhere with respect to the Lebesgue measure.*

The above theorem is a direct consequence of the following.

Theorem 4.8 ([8, Theorem 5.9], see also [1])**.** *Let M be a Möbius manifold. Suppose that M is neither a spherical space form nor a Euclidean space form. Then there is a complete Riemannian metric g on M compatible with the conformal structure of M and of class $C^{1+\mathrm{Lip}}$.*

We say a Riemannian metric g is compatible with the conformal structure if g is invariant under the action of the full group of conformal automorphisms of M as a Möbius manifold.

Notice that the Hopf-Rinow theorem holds for Riemannian metrics of class $C^{1+\mathrm{Lip}}$, and we can speak of the completeness of such metrics.

Proof of Theorem 4.7. Let $\widehat{M}$ be as above. If $\widehat{M}$ is a spherical space form, we have the covering map

$$\pi : S^q \to \widehat{M}.$$

The composition $\Delta \circ \pi$ gives a developing map from S^q to S^q. The essential uniqueness of the developing map shows that π is a diffeomorphism. The same argument

shows that $\widehat{M}$ is diffeomorphic to $\mathbf{R}^q$ when $\widehat{M}$ is a Euclidean space form. Therefore $\widehat{M}$ is neither a spherical space form nor a Euclidean space form.

Then Theorem 4.8 shows that there is a Riemannian metric g of $\widehat{M}$ which is of class $C^{1+\text{Lip}}$ and invariant under the action of the full group of conformal transformations of $\widehat{M}$ as a Möbius manifold. Now it is clear that the natural action of $\pi_1(M)$ on $\widehat{M}$ preserves the conformal structure of $\widehat{M}$.

Thus the metric g is invariant under the action of $\pi_1(M)$, and we complete the proof. $\qquad\square$

We have a good description of the mapping $p : \widetilde{M} \to \widehat{M}$.

Proposition 4.9. *The mapping $p : \widetilde{M} \to \widehat{M}$ is a locally trivial fibration.*

Proof. First we fix a smooth Riemannian metric g_0 on M and let $\widehat{g_T}$ denote the invariant metric on $\widehat{M}$ given in the proof of Theorem 4.7. Then the lift $\widetilde{g}_0$ of g_0 to $\widetilde{M}$ defines an orthogonal distribution $\widetilde{P}$ of $T\widetilde{\mathcal{F}}$.

Noticing that p_* gives an isomorphism from $\widetilde{P}$ to $T\widehat{M}$, we have a $C^{1+\text{Lip}}$ Riemannian metric $\widetilde{g}_T$ on $\widetilde{P}$ which projects down to the metric $\widehat{g_T}$ on $\widehat{M}$. We denote by $\|\cdot\|_{\widetilde{g}_T}$ the norm on $\widetilde{P}$ induced by $\widetilde{g}_T$ and by $\|\cdot\|_{\widehat{g_T}}$ the norm on $\widehat{M}$ induced by $\widehat{g_T}$.

We define $\widetilde{g}_{\widetilde{\mathcal{F}}}$ to be the restriction of $\widetilde{g}_0$ to $T\widetilde{\mathcal{F}}$ and put $\widetilde{g} = \widetilde{g}_{\widetilde{\mathcal{F}}} \oplus \widetilde{g}_T$; then $\widetilde{g}$ projects down to a bundle-like metric g on M with respect to $\mathcal{F}$. The metric g is of class $C^{1+\text{Lip}}$ and complete, hence $\widehat{g}$ is also.

Now we fix a point x_0 in $\widehat{M}$ and choose an open neighbourhood V of x_0 such that there is a positive real number R such that $d(p, q) < R$ holds for any points p and q in V, where d is the distance function on $\widehat{M}$ determined by $\widehat{g_T}$.

Let x and y be points in V. We consider the geodesic $\sigma_{x,y} : [0, 1] \to V$ with respect to $\widehat{g_T}$ joining x and y. Now we fix a point $\widetilde{x}$ in $p^{-1}(x)$ and consider the horizontal lift $\sigma_{x,y}(\widetilde{x})$ of $\sigma_{x,y}$ starting at $\widetilde{x}$. It is clear that such a lift is unique if it exists. So we show the existence.

Since we can always find the lift locally, first we assume that $\sigma_{x,y}(\widetilde{x})$ is defined on $[0, s)$, $s \le 1$. We then have that

$$
\begin{aligned}
\int_0^s \left\| \frac{d\sigma_{x,y}(\widetilde{x})}{dt} \right\|_{\widetilde{g}} dt &= \int_0^s \left\| \frac{d\sigma_{x,y}(\widetilde{x})}{dt} \right\|_{\widetilde{g}_T} dt \\
&= \int_0^s \left\| \frac{d\sigma_{x,y}}{dt} \right\|_{\widehat{g_T}} dt \\
&\le \int_0^1 \left\| \frac{d\sigma_{x,y}}{dt} \right\|_{\widehat{g_T}} dt \\
&\le R.
\end{aligned}
$$

Hence the image of $\sigma_{x,y}(\widetilde{x})$ is contained in K, where K is the closed ball centred at x_0 and of radius R with respect to the metric $\widehat{g}$. Now the completeness of $\widehat{g}$ implies that K is compact.

Thus any horizontal lift $\sigma_{x,y}(\widetilde{x})$ of $\sigma_{x,y}$ stays in a fixed compact set K, and the limit $\lim_{t \to s} \sigma_{x,y}(\widetilde{x})(t)$ exists, say p. Then we can consider the horizontal lift of $\sigma_{x,y}$ near p, which must coincide with $\sigma_{x,y}(\widetilde{x})$ near p. So we can extend $\sigma_{x,y}(\widetilde{x})$ to

$[0, s + \epsilon)$ for some positive real number ϵ. This lift also stays in K, and hence we can extend $\sigma_{x,y}(\widetilde{x})$ to $[0, 1]$.

Finally notice that $\sigma_{x,y}(\widetilde{x})$ is a geodesic with respect to $\widehat{g}$, and we define a mapping $\phi : p^{-1}(x) \times V \to p^{-1}(V) \subset \widetilde{M}$ by

$$\phi(\widetilde{x}, y) = \sigma_{x,y}(\widetilde{x})(1);$$

then ϕ is a continuous map. We show that ϕ is a homeomorphism. In fact, for $\widetilde{z} \in p^{-1}(V)$ we put $z = p(\widetilde{z})$ and consider $\sigma_{z,x}$. Then $\sigma_{z,x}$ lifts to a path $\sigma_{z,x}(\widetilde{z})$. We put $\widetilde{x}_z = \sigma_{z,x}(\widetilde{z})(1)$ and then obtain a mapping $\Psi : p^{-1}(V) \to p^{-1}(x) \times V$ such that $\Psi(\widetilde{z}) = (\widetilde{x}_z, z)$. Clearly Ψ is continuous.

Now consider the path $\sigma_{x,z}(\widetilde{x}_z)$. The uniqueness of the lift implies that this path coincides with the path $t \mapsto \sigma_{z,x}(\widetilde{z})(1 - t)$. Hence $\phi \circ \Psi$ is the identity map.

On the other hand, we suppose that $\widetilde{z} = \phi(\widetilde{x}, y) = \phi(\widetilde{x}', y')$. This is equivalent to saying $\widetilde{z} = \sigma_{x,y}(\widetilde{x})(1) = \sigma_{x,y'}(\widetilde{x}')(1)$. Since the definition of ϕ shows that we have $p \circ \phi(\widetilde{x}, y) = y$, we have $y = y'$. Now we reverse the direction of both paths $\sigma_{x,y}(\widetilde{x})$ and $\sigma_{x,y}(\widetilde{x}')$. Then they are the horizontal lifts of $\sigma_{y,x}$ starting at $\widetilde{z}$. Again the uniqueness shows that these paths coincide, in particular $\widetilde{x} = \widetilde{x}'$. Thus $\Psi \circ \phi$ is the identity.

Consequently Ψ gives a local triviality.   □

As a corollary we have the following, which is shown for complete transversely conformal foliations in Blumenthal [3].

Corollary 4.10. *The leaves of $(M, \mathcal{F})$ have the same universal covering.*   □

In the paper [7] Heitsch and Hurder conjectured that if $(M, \mathcal{F})$ admits a full support measure. Then all the Godbillon-Vey classes [7] are zero. Related to this conjecture, we have the following corollary.

Corollary 4.11. *Let $(M, \mathcal{F})$ be a transversely flat conformal foliation of a closed manifold. Suppose that $(M, \mathcal{F})$ admits a good measure, then all the Godbillon-Vey classes of $(M, \mathcal{F})$ vanish.*

Proof. Since $(M, \mathcal{F})$ is continuously Riemannian, there is a continuous distance function on T which is invariant under the action of the holonomy pseudogroup. Then Theorem 4.3 of [7] shows that all the Godbillon-Vey classes of $(M, \mathcal{F})$ are zero.   □

One can ask how the regularity of the metric can be higher as the regularity of the good measure μ becomes higher. On this line, we have the following.

Recall that a good measure μ defines a Γ-invariant measure, denoted by μ_T, on T. For an integer $r \geq 0$, a good measure μ is called a C^r-volume form if we have $\mu_T = f \cdot \lambda$, where λ denotes the Lebesgue measure restricted to T and f is a positive C^r-function on T.

Proposition 4.12. *Let $(M, \mathcal{F})$ be a transversely flat conformal foliation preserving μ. Let μ be a C^r-volume form. Then we can find an invariant metric g_μ of class C^r.*

Proof. First notice that we have

$$f \circ \gamma(x) \cdot \det \gamma_* = f(x)$$

for any $\gamma \in \Gamma$ and x in the domain of γ. This is because f is continuous and the above equation holds a.e. by virtue of the equation $\gamma^* \mu = \mu$.

Now we denote by g^E the Euclidean metric restricted to T and define

$$g_x = f(x)^{\frac{2}{q}} g^E{}_x,$$

where q is the codimension of $\mathcal{F}$.

Notice that for an orientation preserving element γ of $\mathrm{Conf}(\mathbf{R}^q)$, γ_* is an element of $\mathrm{SO}(q)$ and we have

$$(\gamma^* g^E)_{\gamma x} = (\det \gamma_{*x})^{\frac{2}{q}} g^E{}_x.$$

Then we have

$$\begin{aligned}
(\gamma^* g)_{\gamma x} &= f(\gamma(x))^{\frac{2}{q}} (\gamma^* g^E)_{\gamma x} \\
&= f(\gamma(x))^{\frac{2}{q}} (\det \gamma_{*x})^{\frac{2}{q}} g^E{}_x \\
&= f(x)^{\frac{2}{q}} g^E{}_x \\
&= g_x.
\end{aligned}$$

Thus g is invariant under the action of Γ. □

In general the invariant metric g_T in Theorem 4.7 and g_μ in Proposition 4.12 need not be the same. However, if there is a dense leaf, then g_T and g_μ are essentially the same.

Lemma 4.13. *If $(M, \mathcal{F})$ satisfies the conditions Proposition 4.12, and admits a dense leaf, then there is a positive real number c such that $g_T = c \cdot g_\mu$.*

Proof. We regard the metrics g_T and g_μ as the metrics on T invariant under the action of Γ. Since there is a dense leaf, we can find a point x_0 in T such that Γx_0 is dense in T.

Let c be the positive real number c such that

$$(g_T)_{x_0} = c \cdot (g_\mu)_{x_0}.$$

Then the invariance of the metrics implies that

$$(g_T)_x = c \cdot (g_\mu)_x$$

for $x \in \Gamma x_0$. Since both sides are continuous and Γx_0 is dense in T, the above equation holds on T. □

5. THE TRANSVERSELY SIMILAR CASE

In this section we deal with the case where the holonomy group H is contained in $\mathrm{Sim}(\mathbf{R}^q)$. This is the case when the foliation is transversely similar. We prove Theorem C in this section.

The case where the foliation is transversely similar and of codimension two has already been treated in [6] without the assumption on measures, and there is nothing to prove. But our assumption is slightly weaker than that of [6]. So we assume

the codimension of the foliation is greater than one, and the foliation $\mathcal{F}$ has a good measure μ. We retain the same notation as in the previous sections.

Notice that that if Δ is a homeomorphism onto S^q or $\mathbf{R}^q$, then we have already shown the above theorem. So we assume that Δ is a homeomorphism onto neither S^q nor $\mathbf{R}^q$.

The proof will proceed almost in the same way as in [9] or Fried [5]. But we follow it again since it is the heart of the proof of Theorem C and the very base of the classification of flows in [2].

Definition 5.1. We define

$$
\begin{aligned}
\widehat{M}(\infty) &= \Delta^{-1}(\infty), \\
M(\infty) &= \pi(D^{-1}(\infty)) = \pi(p^{-1}\Delta^{-1}(\infty)), \\
\widehat{M}^* &= \widehat{M} \setminus \widehat{M}(\infty), \text{ and} \\
M^* &= M \setminus M(\infty) = \pi(\widetilde{M} \setminus D^{-1}(\infty)).
\end{aligned}
$$

Lemma 5.2. $M(\infty)$ *is a union of a finite number of compact leaves.*

Proof. Suppose the contrary. Then there is a point p of M where the leaves $\{L_i\}$ of $M(\infty)$ accumulate. (The set $\{L_i\}$ may consist of a single non-compact leaf.) This shows that if we choose any lift $\{\widetilde{L}_i\}$ of $\{L_i\}$, then Ω, the orbit of $\{p(\widetilde{L}_i)\}$ by $\pi_1(M)$, has an accumulating point. Since any element of H fixes infinity, Ω is contained in $\widehat{M}(\infty)$. On the other hand, $\widehat{M}(\infty)$ is discrete because Δ is a local homeomorphism. This yields a contradiction. $\qquad\square$

We write $M(\infty) = \bigcup_{i=1}^{l} L_i$, where each L_i is a compact leaf.

Definition 5.3. A domain $\widehat{U}^*$ of $\widehat{M}^*$ is called a copy of a domain U of $\mathbf{R}^q$ if the restriction of Δ to $\widehat{U}^*$ is a homeomorphism.

Definition 5.4. We say that a closed subset $\widehat{l}^*$ of $\widehat{M}^*$ is a complete half line if for any copy $\widehat{B}^*$ of an open ball B in $\widehat{M}^*$, the image of $\widehat{B}^* \cap \widehat{l}^*$ by Δ is equal to $B \cap l$, where l is a complete half line in $\mathbf{R}^q$.

A complete half line $\widehat{l}^*$ is called short if $l = \Delta(\widehat{l}^*)$ is not a complete half line in $\mathbf{R}^q$. We always parametrize l and $\widehat{l}^*$ as follows.

Let α and β be the endpoints of l such that $\alpha \in l$ and $\beta \notin l$. Then we define

$$
l(t) = \frac{1}{t+1}(\alpha + t\beta)
$$

for $t \in [0, \infty)$, and parametrize $\widehat{l}^*$ so that $\Delta(\widehat{l}^*(t)) = l(t)$ holds.

Lemma 5.5. *For any short complete half line $\widehat{l}^*$, there is a compact saturated neighbourhood U of $M(\infty)$ which is disjoint from $\pi(p^{-1}(\widehat{l}^*))$.*

Proof. For any leaf L_i of $M(\infty)$, there is a compact saturated neighbourhood U_i of L_i which is disjoint from $\pi(p^{-1}\widehat{l}^*(0))$. For, if not, then $\pi(p^{-1}\widehat{l}^*(0))$ must accumulate to L_i. In other words, L_i has a contracting holonomy. This is a contradiction because we have already shown that $(M, \mathcal{F})$ is continuously Riemannian.

Now by shrinking U_i, we may assume that U_i is evenly covered by π, and for each connected component $\widehat{E}_k^*$ of $p(\pi^{-1}(U_i))$, Δ is a homeomorphism from $\widehat{E}_k^*$ onto $E_k = E(a_k, R_k) = \{|x - a_k| \geq R_k\} \cup \{\infty\}$.

Then we have $\widehat{l}^* \cap \widehat{E}_k^* = \phi$. The reason is as follows. Suppose the contrary then $l \cap E_k$ is a half line starting at a point of ∂E_k, where $l = \Delta(\widehat{l}^*)$. Since $\widehat{l}^*$ is short, $p = \lim_{t\to\infty} l(t)$ exists. But for any copy $\widehat{B}^*$ of an open ball B centered at p, its intersection with $\widehat{l}^*$ cannot be mapped to $B \cap l$ by Δ. This contradicts the completeness of $\widehat{l}^*$.

So $p^{-1}(\widehat{l}^*) \cap p^{-1}(\widehat{E}_k^*) = \phi$ for each component $\widehat{E}_k^*$, and consequently we have $\pi(p^{-1}(\widehat{l}^*)) \cap U_i = \phi$. Now we define $U = \bigcup_{i=1}^{l} U_i$; then $\pi(p^{-1}(\widehat{l}^*)) \cap U = \phi$. $\square$

Definition 5.6. For a point $\widehat{x} \in \widehat{M}^*$, we define $r(\widehat{x})$ to be the maximum of the radii of copies of balls centered at $\widehat{x}$.

From now on we will denote by x the image of a point $\widehat{x}$ of $\widehat{M}^*$ under Δ.

Lemma 5.7. *The function r is of finite value.*

Proof. Suppose $r(\widehat{x}) = \infty$; then $\widehat{x}$ is contained in a copy $\widehat{P}$ of $\mathbf{R}^q$. Since we assumed that Δ is not a homeomorphism, we have $\widehat{M}^* \neq \widehat{P}$. So we can find a point $\widehat{y}$ of $\partial\widehat{P}$ and a sequence $\{\widehat{y_n}\}$ of $\widehat{P}$ converging to y.

Since $\Delta(\widehat{y_n})$ tends to ∞, we have $D(\widehat{y}) = \infty$. Therefore there is a neighbourhood Q of $\widehat{y}$ which is mapped to $E(0, R)$ by Δ for some positive real number R. Then Δ is a homeomorphism from $P \cup Q$ to S^q. Since $P \cup Q$ is open and closed in $\widehat{M}$, we see that $\widehat{M} = P \cup Q$. This contradicts our assumption. $\square$

Lemma 5.8. *The function r is continuous.*

Proof. Let $\widehat{x}$ be a point in $\widehat{M}$ and let $\widehat{B}$ be the open ball centred at $\widehat{x}$, of radius $r(\widehat{x})$. We consider the Euclidean metric on $B = \Delta(\widehat{B})$.

Let y be a point in B such that the distance between x and y is less than ϵ. Then the open ball centred at y and of radius $r(\widehat{x}) - \epsilon$ clearly has a copy, so $r(\widehat{y}) > r(\widehat{x}) - \epsilon$. Conversely, the same argument shows that $r(\widehat{y}) > r(\widehat{x}) - \epsilon$. $\square$

Definition 5.9. We define a continuous Riemannian metric $\widehat{g}$ on $\widehat{M}^*$ by

$$\widehat{g}_{\widehat{x}} = \frac{1}{r(\widehat{x})^2}(\Delta^* g_0)_{\widehat{x}},$$

where g_0 denotes the Euclidean metric on $\mathbf{R}^q$.

Remark 1. $\widehat{g}$ is, in a manner of speaking, the transverse Fried metric.

Remark 2. This definition is an analogue of Definition 5.2 of [8]. In the present case the centres of balls are preserved under the action of $\pi_1(M)$ or H, so we can consider normal maximal balls centred at $\widehat{x}$. By using this notion instead of normal maximal balls, we obtain this metric.

The following lemma is clear from the definition.

Lemma 5.10. *$\widehat{g}$ is invariant under the action of $\pi_1(M)$.* $\square$

Hence we have a transverse invariant metric g on M^*.

We define a function $\widehat{d} = \widehat{d}_g : \widehat{M}^* \times \widehat{M}^* \to \mathbf{R}$ by the formula

$$\widehat{d}(p,q) = \inf_{\rho} \int_0^1 \left\| \frac{d\rho}{dt}(t) \right\|_{\widehat{g}} dt,$$

where $\rho : [0,1] \to \widehat{M}^*$ runs through piecewise smooth paths in $\widehat{M}^*$ such that $\rho(0) = p$ and $\rho(1) = q$.

It is easy to see that $\widehat{d}$ is of finite value on $\widehat{M}^* \times \widehat{M}^*$, and is continuous. Since we have $\widehat{d}(\gamma p, \gamma q) = \widehat{d}(p,q)$, we have a continuous function d on $M^* \times M^*$ which locally coincides with $\widehat{d}$.

Hereafter we will describe $\widehat{g}$ more explicitly and make it clear that $\widehat{g}$ is of class C^ω.

Proposition 5.11. *Let $\widehat{B} = \widehat{D}_{r(\widehat{a})}(\widehat{a})$ be the maximal copy of an open ball. Then there is a unique copy of a half space $\widehat{H}$ which contains $\widehat{B}$.*

Proof. First we show the existence. Without loss of generality, we may assume that $r(\widehat{a}) = 1$ and $\Delta(\widehat{a}) = e = (0, \ldots, 0, 1)$. Since Δ restricted to $\widehat{B}$ is a homeomorphism, we may assume

$$\widehat{g}|_{\widehat{B}} = \frac{1}{r(\widehat{x})^2}(g_0)_{\widehat{x}},$$

where $\widehat{B}$ is identified with B.

Now by the maximality of $\widehat{B}$, we can find a radius $\widehat{l}^*$ of $\widehat{B}$ which is a complete half line. We may assume $l = \Delta(\widehat{l}^*) = \{x_1 = \cdots = x_{q-1} = 0, 0 < x_q \le 1\}$. Then the parametrization of l is

$$l(t) = \left(0, \ldots, 0, \frac{1}{t+1}\right),$$

where $t \in [0, \infty)$.

Let $\widehat{x}$ be a point in $\widehat{B}$. We define

$$\theta(\widehat{x}) = \arccos \frac{|g_{0o}(\overrightarrow{ox}, \overrightarrow{oe})|}{\|x\| \cdot \|e\|}$$

to be the angle of $x = \Delta(\widehat{x})$ and e considered as vectors, where o is the origin of $\mathbf{R}^q$. Then we have following sublemmas.

Sublemma 5.12. *We have*

$$\widehat{d}(\widehat{x}, \widehat{l}_q) \ge \theta(x), \quad for \ \widehat{x} \in \widehat{B},$$

where $l_q = \{x_1 = \cdots = x_{q-1} = 0, 0 < x_q < 2\} \subset B$ and $\widehat{l}_q$ is the lift of l_q to $\widehat{B}$.

Proof of Sublemma 5.12. We work in B, and it suffices to show that

$$\overline{d}(x, l_q) \ge \theta(x),$$

where $\overline{d}$ is the metric obtained from $\widehat{d}$ by projecting by the homeomorphism $\Delta|_{\widehat{B}}$.

Let γ be a piecewise smooth path in B from x to a point in l_q. So γ is a mapping from $[0, 1]$ to B.

We denote by $\|\cdot\|$ the Euclidean metric. We write $\gamma(t) = \|\gamma(t)\|\,\varphi(t)$; then we have

$$\frac{d\gamma}{dt}(t) = \frac{d\|\gamma\|}{dt}(t)\varphi(t) + \|\gamma(t)\|\frac{d\varphi}{dt}(t).$$

On the other hand, since $\|\varphi(t)\| \equiv 1$, we have $g_0\left(\varphi(t), \dfrac{d\varphi}{dt}(t)\right) \equiv 0$. So we have

$$\left\|\frac{d\gamma}{dt}(t)\right\|^2 = \left\|\frac{d\|\gamma\|}{dt}(t)\right\|^2 \|\varphi(t)\|^2 + \|\gamma(t)\|^2 \left\|\frac{d\varphi}{dt}(t)\right\|^2$$

$$\geq \|\gamma(t)\|^2 \left\|\frac{d\varphi}{dt}(t)\right\|^2.$$

Now we define

$$\mathrm{length}_{\overline{d}}(\gamma) = \int_0^1 \left\|\frac{d\gamma}{dt}(t)\right\|_{\overline{g}} dt,$$

where $\overline{g}$ denotes the metric obtained from $\widehat{g}$ by projecting by $\Delta|_{\widehat{B}}$.

We claim that $r(\widehat{y}) \leq \|y\|$ holds for any $\widehat{y} \in \widehat{B}$. Suppose the contrary; then the origin of $\mathbf{R}^q$ is contained in $A = B \cup \{x \in \mathbf{R}^q; \|x - y\| < r(\widehat{y})\}$, which has a copy containing $\widehat{a}$. This contradicts the completeness of $\widehat{l}^*$, since the projected image of the intersection of l and any copy of an open ball $B_0 \subset A$ centred at the origin is not of the form $l' \cap B_0$, where l' is a complete half line in $\mathbf{R}^q$. Consequently we have

$$\widehat{g}_{\widehat{x}} \geq \frac{1}{\|x\|^2} g_0.$$

It follows that we have

$$\mathrm{length}_{\overline{d}}(\gamma) \geq \int_0^1 \frac{\left\|\dfrac{d\gamma}{dt}(t)\right\|}{\|\gamma(t)\|} dt$$

$$\geq \int_0^1 \left\|\frac{d\varphi}{dt}(t)\right\| dt$$

$$\geq \theta(x).$$

This implies $\overline{d}(x, l_q) \geq \theta(x)$.   □

Sublemma 5.13. *If $\widehat{y} \in \widehat{M}^*$ is not contained in $\widehat{B}$, then there is a positive real number ϵ_0 such that $\widehat{d}(\widehat{y}, \widehat{l}^*) > \epsilon_0$ holds.*

Proof of Sublemma 5.13. We identify $\widehat{B}$ and B, and define

$$C_\varphi = \{\widehat{x} \in \widehat{B}; \theta(\widehat{x}) = \varphi\},$$

$$B_l = \{\widehat{x} = (x_1, \ldots, x_q) \in \widehat{B}; \left\|\overrightarrow{e\widehat{x}}\right\| = \frac{4}{3}, x_q > 1\},$$

$$B_m = \{\widehat{x} = (x_1, \ldots, x_q) \in \widehat{B}; \frac{4}{3} \leq \left\|\overrightarrow{e\widehat{x}}\right\| \leq \frac{5}{3}, x_q > 1\},$$

$$B_h = \{\widehat{x} = (x_1, \ldots, x_q) \in \widehat{B}; \left\|\overrightarrow{e\widehat{x}}\right\| = \frac{5}{3}, x_q > 1\},$$

where e denotes the centre of $\widehat{B}$. By choosing φ sufficiently small, we may assume that $B_h \cap C_\varphi \neq \phi$ and $B_l \cap C_\varphi \neq \phi$.

Let γ be a piecewise smooth path in $\widehat{M}^*$ from $\widehat{y}$ to a point in $\widehat{l}^*$. Since $\widehat{y}$ is not contained in $\widehat{B}$, we have the following possibilities:

1) γ intersects both B_l and B_h, or

2) γ intersects C_φ.

If we have case 2), then by Sublemma 5.12 we have $\text{length}_{\widehat{d}}(\gamma) > \varphi$.

Now we assume that γ does not intersect C_φ, so we have case 1). Since γ does not intersect C_φ, we may assume $\gamma[t_0, t_1] = \gamma[0,1] \cap B_m$. Recall that we have $r(\widehat{y}) \leq \|y\|$ for any $\widehat{y} \in \widehat{B}$; then

$$\text{length}_{\widehat{d}}(\gamma) > \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt}(t) \right\|_{\widehat{g}} dt$$

$$\geq \int_{t_0}^{t_1} \left\| \frac{d\|\gamma\|}{dt}(t) \right\|_{\widehat{g}} dt$$

$$\geq \int_{t_0}^{t_1} \frac{3}{5} \left\| \frac{d\|\gamma\|}{dt}(t) \right\| dt$$

$$\geq \frac{1}{5}.$$

If we put $\epsilon_0 = \min\{\frac{1}{5}, \varphi\}$, then we have $\widehat{d}(\widehat{y}, \widehat{l}^*) > \epsilon_0$.   □

Now we return to prove Proposition 5.11. Notice that $\widetilde{L}_t = p^{-1}\widehat{l}^*(t)$ is a single leaf of $\widetilde{\mathcal{F}}$. By Lemma 5.5 we have a compact foliated manifold $M_R = M \setminus \text{Int} \, U$ which contains $\pi(p^{-1}(\widehat{l}^*))$, where $\text{Int} \, U$ denotes the interior of U. Since M_R is compact, we can choose an increasing sequence $\{t_i\}$ of positive real numbers such that if we choose points c_i of $\widetilde{L}_{t_i}$ appropriately, then $\{\pi(c_i)\}$ converges to a point c in M_R. Let L_c be the leaf of $\mathcal{F}$ which contains c, and $\widetilde{L}_c$ a lift of L_c. Now we assume $d(\pi(c_1), c)$ is small enough; then by Sublemma 5.13 we can find a point $\widehat{b} \in \widehat{B}$ which satisfies $p(\widetilde{L}_c) \in \pi_1(M)\widehat{b}$. So we may assume that $\widehat{b} = p(\widetilde{L}_c) \in \widehat{B}$.

Then there is a sequence $\{\gamma_i\}$ of $\pi_1(M)$ such that $\widehat{d}(\widehat{b}, \gamma_i^{-1}(\widehat{l}^*(t_i)))$ converges to zero. Since $\pi_1(M)$ acts on $\widehat{M}^*$ as an isometry, we see that $\widehat{d}(\gamma_i(\widehat{b}), \widehat{l}^*(t_i))$ converges to zero. Then we may assume by virtue of Sublemma 5.13 that $\gamma_i(\widehat{b}) \in \widehat{B}$.

Now we pass to the model B, and take elements $f_i = \phi(\gamma_i)$ of $\text{Sim}(\mathbf{R}^q)$ and $b = D(L_c) \in B$. Then by passing to a subsequence if necessary, we have

1) $f_i(b) \in B$.

2) $\theta(f_i(b))$ converges to 0.

3) $f_i(b)$ converges to $0 \in \mathbf{R}^q$.

For an element f of $\text{Sim}(\mathbf{R}^q)$, we write $f(x) = rAx + v$, where $r > 0$, $A \in \text{SO}(q)$, and $v \in \mathbf{R}^q$. Then we define $|f| = r$, and $P(f) = A$. Now we may assume

4) $P(f_i)$ converges to an element P_0 of $\text{SO}(q)$.

Noticing that

$$|f_i| = \frac{r(f_i(b))}{r(\widehat{b})} \leq \frac{\|f_i(b)\|}{r(\widehat{b})},$$

we have

5) $|f_i|$ converges to 0.

Furthermore we may assume, by taking i and j sufficiently large, that $(j \gg i \gg 1)$

   6) $P(f_i f_j^{-1})$ is sufficiently near the identity element id of $SO(q)$.

   7) $\left| f_j f_i^{-1} \right| = |f_j| / |f_i|$ is sufficiently near 0.

Now by 2), $\overrightarrow{0 f_j(b)}$ is almost parallel to e, and hence almost perpendicular to ∂B. Then by 6) and the fact that $f_i f_j^{-1}$ is a Euclidean similarity transformation, we see that

   8) $\overrightarrow{f_i f_j^{-1}(0) f_i(b)}$ is almost parallel to e and almost perpendicular to $f_i f_j^{-1}(\partial B)$.

Notice that since $f_i f_j^{-1}(f_j(b)) \in B$, $\gamma_i \gamma_j^{-1} B^* \cap B^*$ is nonempty. Moreover, by the similarity, $f_i f_j^{-1} B \cap B$ is connected, hence $\gamma_i \gamma_j^{-1} B^* \cap B^*$ is a copy of $f_i f_j^{-1} B \cap B$. So $\gamma_i \gamma_j^{-1} B^* \cup B^*$ is a copy of $f_i f_j^{-1} B \cup B$. Then the completeness of l shows that

   9) $f_i f_j^{-1}(0) \notin B$, and

  10) $f_i f_j^{-1}(B) \not\ni 0$.

We define δ_j to be the Euclidean distance between $f_j(b)$ and ∂B. Then condition 2) implies that $\delta_j / \|f_j(b)\|$ converges to 1. Notice that the open ball $D_{\delta_j}(f_j(b))$ centred at $f_j(b)$, of radius δ_j, is contained in B.

Let ϵ be a positive real number less than 1. Then we may assume that we have $\delta_i / \|f_i(b)\| > 1 - \epsilon/2$, and $\|f_i(b)\| < \epsilon$.

Now we claim that $\left\| f_i f_j^{-1}(0) \right\| < 3\epsilon$ holds for any integers i and j. Suppose the contrary; then we can find integers i and j such that $\left\| f_i f_j^{-1}(0) \right\| \geq 3\epsilon$. Then we have

$$
\begin{aligned}
\left| f_i f_j^{-1} \right| &= \frac{\left\| \overrightarrow{f_i f_j^{-1}(0) f_i(b)} \right\|}{\left\| \overrightarrow{0 f_j(b)} \right\|} \cdot \frac{\left\| \overrightarrow{0 f_i f_j^{-1}(0)} \right\| - \left\| \overrightarrow{0 f_i(b)} \right\|}{\left\| \overrightarrow{0 f_j(b)} \right\|} \\
&> \frac{3\epsilon - \epsilon}{\left\| \overrightarrow{0 f_j(b)} \right\|} = \frac{2\epsilon}{\|f_j(b)\|}.
\end{aligned}
$$

On the other hand, since $D_{\delta_j}(f_j(b))$ is contained in B, the ball

$$
f_i f_j^{-1}(D_{\delta_j}(f_j(b))) = D_{\left| f_i f_j^{-1} \right| \delta_j}(f_i(b))
$$

is contained in $f_i f_j^{-1}(B)$.

Now we have, however,

$$
\left| f_i f_j^{-1} \right| \delta_j > 2\epsilon \frac{\delta_j}{\|f_j(b)\|} > 2\epsilon \left(1 - \frac{\epsilon}{2}\right) = 2\epsilon - \epsilon^2 > \epsilon.
$$

This shows that the origin is contained in $D_{\left| f_i f_j^{-1} \right| \delta_j}(f_i(b))$, because $\|f_i(b)\| < \epsilon$. Hence $f_i f_j^{-1}(B)$ contains the origin. This contradicts property 10). Thus the claim is proved.

Consequently we may assume $d(0, f_i f_j^{-1}(0))$ is very small. Now 7), 8), 9), and 10) show that $f_i f_j^{-1}(\partial B)$ tends to the hyperplane $\{x \in \mathbf{R}^q; \ x_q = 0\}$. This shows that for any $x \in \{x_q > 0\}$, we can find integers $j \gg i \gg 1$ such that $f_i f_j^{-1} B \ni x$. Thus the union $\bigcup_{j \gg i \gg 1} f_i f_j^{-1} B \cup B$ contains a half space H which contains B. Since $f_i f_j^{-1} B \cup B$ has a copy containing $\widehat{a}$, H also has a copy $\widehat{H}$ containing $\widehat{B}$.

Now we show the uniqueness. Let $\widehat{H}$ be a copy of a half space containing $\widehat{B}$. It is clear that $H = \Delta(\widehat{H})$ is tangent to $\Delta(\widehat{B}) = D_{r(a)}(a)$ and the radius to the point of the tangency is the developing image of a short complete line. That is, there is a unique short complete line in $\widehat{M}^*$ which is contained in $\widehat{B}$. This shows the uniqueness of $\widehat{H}$. $\qquad\square$

Definition 5.14. We denote by $\widehat{H}(\widehat{a})$ the unique copy of the half space as in Proposition 5.11. We put $H(\widehat{a}) = \Delta(\widehat{H}(\widehat{a}))$. Finally we denote by $\tau(\widehat{a})$ the point of tangency of $H(\widehat{a})$ and $D_{r(a)}(a)$.

Hereafter we use the same notations and conventions as in the proof of Proposition 5.11. Notice then that $\tau(\widehat{a}) = 0$.

Since Δ is a local homeomorphism, it is injective on the closure $\overline{\widehat{H}(\widehat{a})}$ of $\widehat{H}(\widehat{a})$ in $\widehat{M}^*$, and $\Delta(\partial\widehat{H}(\widehat{a}))$ is an open set of $\partial H(\widehat{a})$, the frontier of $H(\widehat{a})$ in $\mathbf{R}^q$.

Definition 5.15. We define $L(\widehat{a}) = \partial H(\widehat{a}) \setminus \Delta(\partial\widehat{H}(\widehat{a}))$.

Here we immediately see that $x \in L(\widehat{a})$ if and only if there is a short complete line $\widehat{l}^*$ such that $\widehat{l}^*(0) = \widehat{a}$ and $\lim_{t\to\infty} l(t) = x$. For any point y in $\partial H(\widehat{a}) \setminus L(\widehat{a})$, we denote by $\widehat{y}$ the unique point in $\partial\widehat{H}(\widehat{a})$ such that $\Delta(\widehat{y}) = y$.

Lemma 5.16. *For any point x in $\overline{H(\widehat{a})} \setminus L(\widehat{a})$, $\partial H(\widehat{x})$ passes through $\tau(\widehat{a})$, where $\widehat{x}$ is the lift of x in $\widehat{H}(\widehat{a})$.*

Proof. First we choose a point $\widehat{y}$ near $\widehat{x}$ contained in $\widehat{H}(\widehat{a})$. Then $\widehat{y} \in \widehat{H}(\widehat{a}) \cap \widehat{H}(\widehat{x})$. So $H(\widehat{a}) \cup H(\widehat{x})$ has a copy in $\widehat{M}^*$. Hence $\tau(\widehat{a}) \notin H(\widehat{x})$.

Now we suppose that $\tau(\widehat{a}) \notin \partial H(\widehat{x})$. We consider the transformation $f_j f_i^{-1} = (f_i f_j^{-1})^{-1}$. Recall that $\left|f_j f_i^{-1}\right|$ is so small that $f_j f_i^{-1}$ acts as a strong contraction; then since the points $f_i(b)$, $f_j(b)$ and $\tau(\widehat{a})$ are sufficiently near, it follows by the contraction principle that $f_j f_i^{-1}$ has a fixed point near $\tau(\widehat{a})$.

On the other hand, $P(f_j f_i^{-1})$ is sufficiently near the identity. Consequently $f_j f_i^{-1} \partial H(\widehat{x})$ is almost parallel to $\partial H(\widehat{x})$ and nearer to $\tau(\widehat{a})$.

Now we may assume that $f_j f_i^{-1}(\Delta(\widehat{y})) \in H(\widehat{a})$, since $P(f_j f_i^{-1})$ is near the identity. Hence we see that $\widehat{H}(\widehat{a}) \cap \gamma_j \gamma_i^{-1} \widehat{H}(\widehat{x}) \neq \phi$. Therefore $H(\widehat{a}) \cup H(\widehat{x}) \cup f_j f_i^{-1} H(\widehat{x})$ has a copy in $\widehat{M}$. This contradicts the fact that $\tau(\widehat{x}) \in L(\widehat{x})$. $\qquad\square$

Lemma 5.17. *$L(\widehat{a})$ is an Affine subspace of $\mathbf{R}^q$.*

Proof. Let x, y in $L(\widehat{a})$. We can write $x = \tau(\widehat{c})$ for some $\widehat{c} \in H(\widehat{a})$, and $y = \tau(\widehat{d})$. Notice that $L(\widehat{a}) = L(\widehat{c}) = L(\widehat{d})$. Suppose that there is a point b on the line passing through x and y and which does not belong to $L(\widehat{a})$. Since b is in $\partial H(\widehat{a})$, we can apply Lemma 5.16 to b and see that $\partial H(\widehat{b})$ passes through $\tau(\widehat{c}) = x$ and $\tau(\widehat{d}) = y$. So $\partial H(\widehat{b})$ must pass through b. This is absurd. $\qquad\square$

Lemma 5.18. *The correspondence $\widehat{a} \mapsto L(\widehat{a})$ is locally constant.*

Proof. For any point $\widehat{b}$ of $\widehat{H}(\widehat{a})$, Lemma 5.16 shows that $\partial H(\widehat{b})$ passes through $\tau(\widehat{a})$, and $\tau(\widehat{a}) \in L(\widehat{b})$. Since $L(\widehat{b}) \cap H(\widehat{a}) = \phi$ and $\tau(\widehat{a}) \in L(\widehat{b}) \cap \partial H(\widehat{a})$, we have $L(\widehat{b}) \subset \partial H(\widehat{a}) \cap \partial H(\widehat{b})$. Similarly we have $L(\widehat{a}) \subset \partial H(\widehat{a}) \cap \partial H(\widehat{b})$.

Notice that $H(\widehat{a}) \cup H(\widehat{b})$ has a copy since $\widehat{b} \in \widehat{H}(\widehat{a}) \cap \widehat{H}(\widehat{b})$. Now let $x \in L(\widehat{a}) \setminus L(\widehat{b})$; then the fact that $x \in L(\widehat{a})$ shows that $x \notin \widehat{M}^*$. On the other hand, the fact that $x \notin L(\widehat{b})$ shows that $x \in \widehat{M}^*$.

This is impossible, and hence $L(\widehat{a}) \subset L(\widehat{b})$. The same argument shows that $L(\widehat{a}) \supset L(\widehat{b})$. Thus we obtain the result. $\square$

Since $\widehat{M}$ is connected, $L(\widehat{a})$ is independent of the choice of $\widehat{a} \in \widehat{M}$. So we write $L_\infty = L(\widehat{a})$. We may assume that L_∞ is a vector subspace of $\mathbf{R}^q$.

Now we show that the image of Δ does not contain the point ∞.

Proposition 5.19. *The mapping Δ is a covering map onto a component of $\mathbf{R}^q \setminus L_\infty$.*

Proof. It is clear that the image of $\widehat{M}$ by Δ is contained in $\mathbf{R}^q \setminus L_\infty \cup \{\infty\}$. The above lemmas show that $\Delta|_{\widehat{M}^*} : \widehat{M}^* \to \mathbf{R}^q \setminus L_\infty$ is a covering map. So it suffices to show that $\Delta^{-1}(\infty) = \phi$. This is clear in the case where $\dim_\mathbf{R} L_\infty > 0$, because $\infty \in \overline{L_\infty}$.

Now suppose that $\dim_\mathbf{R} L_\infty = 0$, i.e. $L_\infty = \{0\}$, and $\Delta^{-1}(\infty) \neq \phi$. If $q > 2$, then $\mathbf{R}^q \setminus \{0\}$ is simply connected and hence $\Delta|_{\widehat{M}^*}$ is a homeomorphism. Thus $\widehat{M}$ is homeomorphic to $\widehat{\mathbf{R}^q} \setminus \{0\}$. This contradicts our assumption.

We assume $q = 2$; then $\widehat{M}^*$ is a covering of $\mathbf{R}^2 \setminus \{0\}$ with covering degree n. If $n = 0$, then the same argument shows that $\widehat{M} \cong \mathbf{R}^2$, which is a contradiction. If we have $n \neq 0$, then it is clear that Δ cannot be a local homeomorphism near the point ∞. This is again a contradiction. $\square$

Proof of Theorem C. Let $\dim_\mathbf{R} L_\infty = q_0$ and consider $\mathbf{R}^q = \mathbf{R}^{q_0} \times \mathbf{R}^{q_1}$, where $q_1 = q - q_0$. When going down to $\mathbf{R}^q$, $\overline{g}$ is given by

$$\overline{g}_x = \frac{1}{\|x_2\|^2} g_{0x}, \text{ where } x = (x_1, x_2) \in \mathbf{R}^{q_0} \times \mathbf{R}^{q_1}.$$

This proves the theorem. $\square$

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3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan
E-mail address: asuke@ms.u-tokyo.ac.jp